Parametrized Curves

Definition A parametrized differentiable curve is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

Note that if $\alpha : I \to \mathbb{R}^3$ is given by $\alpha(t) = (x(t), y(t), z(t))$ for $t \in I$, then α is a differentiable curve if and only if $x(t)$, $y(t)$, $z(t)$ are differentiable (or smooth) functions on I. If $\alpha : I \to \mathbb{R}^3$ is differentiable, the vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the tangent vector (or velocity vector) of the curve α at t.

Example The helix $\alpha(t) = (a \cos t, a \sin t, bt)$, $t \in \mathbb{R}$, is a differentiable curve on the cylinder $x^2 + y^2 = a^2$. Note that $\alpha'(t) = (-a \sin t, a \cos t, b) \neq (0, 0, 0)$, for all $t \in \mathbb{R}$, i.e. the tangent vector of α is not a zero vector for all t.

Example $\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$ is a differentiable plane curve. Note that $\alpha'(0) = (0, 0)$, i.e. the tangent vector is a zero vector at $t = 0$.

Definition A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0 \in \mathbb{R}^3$ for all $t \in I$.

Definition Let $\alpha : I \to \mathbb{R}^3$, be a regular parametrized curve. Given $t_0 \in I$, define the arc length function of α from the point t_0 by

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|\, du, \quad \text{where} \quad |\alpha'(u)| = \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} = \text{ length of } \alpha'(u).$$

Remarks

1. Since $\dfrac{ds}{dt} = |\alpha'(t)| \neq 0$ for all $t \in I$, $s = s(t)$ has a differentiable inverse $t = t(s)$ with $\dfrac{dt}{ds} = \dfrac{1}{ds/dt}$.
2. If the parameter t is already the arc length measured from some point, then $\dfrac{ds}{dt} = 1 = |\alpha'(t)|$, i.e. the velocity vector has constant length equal to 1.
   Conversely, if $|\alpha'(t)| = 1$ for all t, then
   $$s = \int_{t_0}^{t} du = t - t_0; \text{ i.e., } t \text{ is the arc length of } \alpha \text{ measured from some point.}$$
3. To simplify our exposition, we shall restrict ourselves to curves parametrized by arc length; we shall see later (see Sec. 1-5) that this restriction is not essential. In general, it is not necessary to mention the origin of the arc length s, since most concepts are defined only in terms of the derivatives of $\alpha(s)$.
4. It is convenient to set still another convention. Given the curve α parametrized by arc length $s \in (a, b)$, we may consider the curve β defined in (a, b) by $\beta(s) = \alpha(-(s - a) + b)$, which has the same trace as the first one but is described in the opposite direction. We say, then, that these two curves differ by a change of orientation.

The Vector Product in $\mathbb{R}^3$

Definition Two ordered bases $e = \{e_i\}$ and $f = \{f_i\}$, $i = 1, \ldots, n$, of an n-dimensional vector space V have the same orientation if the matrix of change of basis has positive determinant. We denote this relation by $e \sim f$.

Remark From elementary properties of determinants, it follows that $e \sim f$ is an equivalence relation; i.e., it satisfies

1. $e \sim e$. [Since $e = Ie$ and $\det I = 1 > 0$, where I is the $n \times n$ identity matrix.]
2. If $e \sim f$, then $f \sim e$. [If $f = Ae$, then $e = A^{-1}f$ and $\det A^{-1} = \dfrac{1}{\det A} > 0$.]
3. If $e \sim f$ and $f \sim g$, then $e \sim g$. [If $f = Ae$ and $g = Bf$, then $g = BAe$ and $\det(BA) = \det B \det A > 0$.]

The set of all ordered bases of V is thus decomposed into equivalence classes (the elements of a given class are related by $\sim$) which by property 3 are disjoint. Since the determinant of a change of basis is either positive or negative, there are only two such classes.

Each of the equivalence classes determined by the above relation is called an orientation of V. Therefore, V has two orientations, and if we fix one of them arbitrarily, the other one is called the opposite orientation.

In the case $V = \mathbb{R}^3$, there exists a natural ordered basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and we shall call the orientation corresponding to this basis the positive orientation of $\mathbb{R}^3$, the other one being the negative orientation (of course, this applies equally well to any $\mathbb{R}^n$). We also say that a given ordered basis of $\mathbb{R}^3$ is positive (or negative) if it belongs to the positive (or negative) orientation of $\mathbb{R}^3$. Thus, the ordered basis e_1, e_3, e_2 is a negative basis, since the matrix which changes this basis into e_1, e_2, e_3 has determinant equal to -1.

Definition Let $u, v \in \mathbb{R}^3$. The vector product of $u = \sum_{i=1}^{3} u_i e_i$ and $v = \sum_{i=1}^{3} v_i e_i$ (in that order) is the unique vector $u \wedge v \in \mathbb{R}^3$ characterized by

$$(u \wedge v) \cdot w = \det(u, v, w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad \text{for all } w = \sum_{i=1}^{3} w_i e_i \in \mathbb{R}^3,$$

where $|a_{ij}|$ denotes the determinant of the matrix (a_{ij}). It is immediate from the definition that

$$u \wedge v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3.$$

Remarks

(a) It is also very frequent to write $u \wedge v$ as $u \times v$ and refer to it as the cross product.

The following properties can easily be checked (actually they just express the usual properties of determinants):

1. $u \wedge v = -v \wedge u$ (anticommutativity).
2. $u \wedge v$ depends linearly on u and v; i.e., for any real numbers a, b, we have
$$(au + bw) \wedge v = au \wedge v + bw \wedge v.$$
3. $u \wedge v = 0$ if and only if u and v are linearly dependent.
4. $(u \wedge v) \cdot u = 0$ and $(u \wedge v) \cdot v = 0$.

(b)
- It follows from (a) property 4 that the vector product $u \wedge v$ is normal to a plane generated by u and v.

- If $u \wedge v \neq 0 \in \mathbb{R}^3$, then $\det(u, v, u \wedge v) = (u \wedge v) \cdot (u \wedge v) = |u \wedge v|^2 > 0$. This implies that $\{u, v, u \wedge v\}$ is a positive basis.
- Since
$$(e_i \wedge e_j) \cdot (e_k \wedge e_\ell) = \begin{vmatrix} e_i \cdot e_k & e_j \cdot e_k \\ e_i \cdot e_\ell & e_j \cdot e_\ell \end{vmatrix} \quad \text{for all } i,\ j,\ k,\ \ell = 1,\ 2,\ 3,$$
we have the foolowing identity
$$(u \wedge v) \cdot (x \wedge y) = \begin{vmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{vmatrix} \quad \text{for all } u,\ v,\ x,\ y \in \mathbb{R}^3$$
by observing that both sides are linear in u, v, x, y.
- It follows that
$$|u \wedge v|^2 = (u \wedge v) \cdot (u \wedge v) = \begin{vmatrix} u \cdot u & v \cdot u \\ u \cdot v & v \cdot v \end{vmatrix} = |u|^2 |v|^2 (1 - \cos^2\theta) = A^2,$$
where θ is the angle of u and v, and A is the area of the parallelogram generated by u and v.

Therefore, the vector product of u and v is a vector $u \wedge v$ perpendicular to a plane spanned by u and v, with a norm equal to the area of the parallelogram generated by u and v and a direction such that $\{u, v, u \wedge v\}$ is a positive basis.

(c) Since
$$(e_i \wedge e_j) \wedge e_k = (e_i \cdot e_k)e_j - (e_j \cdot e_k)e_i \quad \text{for all } i,\ j,\ k = 1,\ 2,\ 3,$$
we have the following identity:
$$(u \wedge v) \wedge w = (u \cdot w)v - (v \cdot w)u,$$
by observing that both sides are linear in u, v, w. In particular, since
$$(e_1 \wedge e_2) \wedge e_2 = (e_1 \cdot e_2)e_2 - (e_2 \cdot e_2)e_1 = -e_1 \neq 0 = e_1 \wedge (e_2 \wedge e_2),$$
the vector product is not associative.

(d) Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) to $\mathbb{R}^3$, $t \in (a, b)$. It follows that $u(t) \wedge v(t)$ is also differentiable and that
$$\frac{d}{dt}(u(t) \wedge v(t)) = \frac{du}{dt} \wedge v(t) + u(t) \wedge \frac{dv}{dt}.$$

The Local Theory of Curves Parametrized by Arc Length

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s, i.e. the tangent vector $\alpha'(s)$ has unit length for all s. Since $\alpha'(s) \cdot \alpha''(s) = 0$, $\alpha''(s)$ is perpendicular to the tangent when $\alpha''(s) \neq 0$ and the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s. $|\alpha''(s)|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s. This suggests the following definition.

Definition Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $|\alpha''(s)| = k(s)$ is called the curvature of α at s.

Example If α is a straight line, $\alpha(s) = us + v$, where u and v are constant vectors with $|u| = 1$, then $k(s) = 0$ for all $s \in \mathbb{R}$. Conversely, if $k(s) = |\alpha''(s)| = 0$ for all $s \in \mathbb{R}$, then by integration $\alpha(s) = us + v$, and the curve is a straight line.

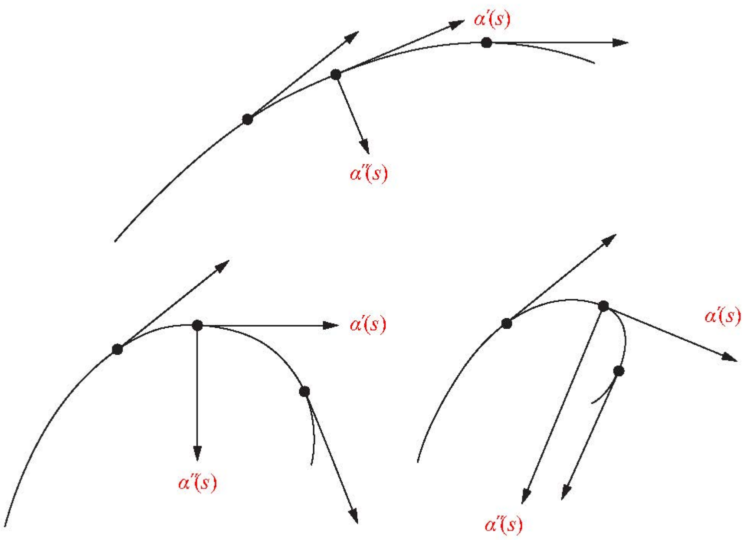
Remarks

- Notice that by a change of orientation, the tangent vector changes its direction; that is, if $\beta(s) = \alpha(-(s-a)+b)$, then

$$\beta'(s) = -\alpha'(-(s-a)+b) \quad \text{and} \quad \beta''(s) = \alpha''(-(s-a)+b),$$

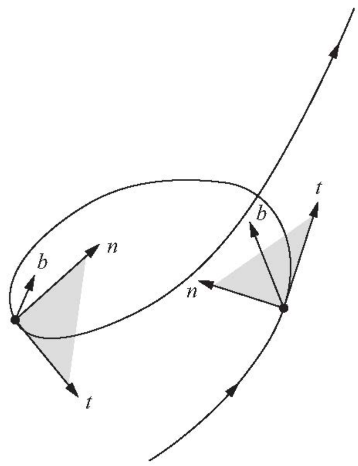
  i.e., $\alpha''(s)$ and the curvature remain invariant under a change of orientation.

- At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$. Moreover, $\alpha''(s)$ is normal to $\alpha'(s)$, because by differentiating $\alpha'(s) \cdot \alpha'(s) = 1$ we obtain $\alpha''(s) \cdot \alpha'(s) = 0$. Thus, $n(s)$ is normal to $\alpha'(s)$ and is called the (principal) normal vector at s. The plane determined by the unit tangent and normal vectors, $\alpha'(s)$ and $n(s)$, is called the osculating plane at s.



In what follows, we shall restrict ourselves to curves parametrized by arc length without singular points of order 1, i.e. $\alpha''(s) \neq 0$ for all s. We shall denote by $t(s) = \alpha'(s)$ the unit tangent vector of α at s. Thus, $t'(s) = k(s)n(s)$.

- The unit vector $b(s) = t(s) \wedge n(s)$ normal to the osculating plane and will be called the binormal vector at s. Since $b(s)$ is a unit vector, the length $|b'(s)|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is, $|b'(s)|$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.



To compute $b'(s)$ we observe that, on the one hand, $b'(s)$ is normal to $b(s)$ and that, on the other hand,

$$b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s) = t(s) \wedge n'(s)$$

that is, $b'(s)$ is normal to $t(s)$. It follows that $b'(s)$ is parallel to $n(s)$, and we may write

$$b'(s) = \tau(s) n(s) \quad \text{for some function } \tau(s).$$

Definition Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0$ for each $s \in I$. The number $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the torsion of α at s.

Note that by changing orientation the binormal vector changes sign, since $b(s) = t(s) \wedge n(s)$. It follows that $b'(s)$, and, therefore, the torsion, remain invariant under a change of orientation.

Example If α is a plane curve (that is, $\alpha(I)$ is contained in a plane), then the plane of the curve agrees with the osculating plane; hence, $\tau(s) = 0$ for all $s \in I$.

Conversely, if $\tau(s) = 0$ (and $k(s) \neq 0$) for each $s \in I$, we have $b(s) = b_0 =$ constant vector, and therefore

$$(\alpha(s) \cdot b_0)' = \alpha'(s) \cdot b_0 = 0 \quad \text{for all } s \in I \implies \alpha(s) \cdot b_0 = \text{ constant} \quad \text{for all } s \in I.$$

Hence, $\alpha(s)$ is contained in a plane normal to b_0. The condition that $k(s) \neq 0$ everywhere is essential here. For example, consider the regular curve

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0,0,0), & t = 0 \end{cases} \implies \alpha'(t) = \begin{cases} (1, 0, \frac{2}{t^3} e^{-1/t^2}), & t > 0 \\ (1, \frac{2}{t^3} e^{-1/t^2}, 0), & t < 0 \\ (1,0,0), & t = 0 \end{cases}$$

Using the Sec. 1-5 Exercises 12, one can show that $k(0) = 0$ and the torsion $\tau \equiv 0$ even though α is not a plane curve.

Definition Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0$ for each $s \in I$. To each value of the parameter s, we have associated three orthogonal unit vectors $t(s)$, $n(s)$, $b(s)$, called the Frenet trihedron at s, determined by the Frenet formulas

$$\begin{aligned} t'(s) &= k(s)\, n(s), \\ n'(s) &= -k(s)\, t(s) - \tau(s)\, b(s), \\ b'(s) &= \tau(s)\, n(s). \end{aligned}$$

Example If $\alpha : I \to \mathbb{R}^3$ is a curve parametrized by arc length s satisfying that $\alpha''(s) \neq 0$ for each $s \in I$, then $t(s) = \alpha'(s)$, $n(s) = \dfrac{\alpha''(s)}{|\alpha''(s)|}$, $b(s) = \dfrac{\alpha'(s) \wedge \alpha''(s)}{|\alpha''(s)|}$ and $n'(s) = \dfrac{\alpha'''(s)}{|\alpha''(s)|} - \dfrac{\alpha''(s)\langle \alpha''(s), \alpha'''(s)\rangle}{|\alpha''(s)|^3} \implies \tau(s) = -\langle n'(s), b(s)\rangle = -\dfrac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s)\rangle}{|\alpha''(s)|^2}$.

Example If $\alpha : I \to \mathbb{R}^3$ is a regular curve parametrized by u and if $s = s(u)$ is an arc length parameter, then $\dfrac{du}{ds} = \dfrac{1}{|\alpha'(u)|}$, $\dfrac{d^2u}{ds^2} = \dfrac{d}{du}\left(\dfrac{du}{ds}\right) \cdot \dfrac{du}{ds} = -\dfrac{\langle \alpha'(u), \alpha''(u)\rangle}{|\alpha'(u)|^4}$, and

$$k(u) = \left|\frac{d^2\alpha}{ds^2} \wedge \frac{d\alpha}{ds}\right| = \left|\left[\alpha''(u)\,(\frac{du}{ds})^2 + \alpha'(u)\,\frac{d^2u}{ds^2}\right] \wedge \alpha'(u)\,\frac{du}{ds}\right| = \frac{|\alpha'(u) \wedge \alpha''(u)|}{|\alpha'(u)|^3},$$

$$\tau(s) = -\frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s)\rangle}{|\alpha''(s)|^2} = -\frac{\langle (\frac{du}{ds})^3 \cdot \alpha'(u) \wedge \alpha''(u), \alpha'''(u)\,(\frac{du}{ds})^3 + 3\alpha''(u)\,\frac{du}{ds}\,\frac{d^2u}{ds^2} + \alpha'(u)\,\frac{d^3u}{ds^3}\rangle}{|\alpha'(u) \wedge \alpha''(u)|^2/|\alpha'(u)|^6}$$
$$= -\frac{\langle \alpha'(u) \wedge \alpha''(u), \alpha'''(u)\rangle}{|\alpha'(u) \wedge \alpha''(u)|^2}.$$

Remarks

- In this context, the points $\alpha(s)$ where $\alpha'(s) = 0$ are called a singular point of order 0 and the points $\alpha(s)$ where $\alpha''(s) = 0$ are called singular points of order 1.
- If $\alpha : I \to \mathbb{R}^3$ is a curve parametrized by arc length s such that $\alpha''(s) \neq 0$ for each $s \in I$, then $R = \dfrac{1}{k(s)} = \dfrac{1}{|\alpha''(s)|}$ is called the radius of curvature at s. Of course, one can easily show that a circle of radius r has radius of curvature equal to r.
- Physically, we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to conjecture the following statement, which, roughly speaking, shows that k and τ describe completely the local behavior of the curve.

Fundamental Theorem of The Local Theory of Curves Given differentiable functions $k(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$, satisfying the same conditions, differs from α by a rigid motion; that is, there exists an orthogonal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector $c \in \mathbb{R}^3$ such that $\bar{\alpha} = \rho \circ \alpha + c$.

Proof of Uniqueness Assume that two curves $\alpha = \alpha(s)$ and $\bar{\alpha} = \bar{\alpha}(s)$ satisfy the conditions $k(s) = \bar{k}(s)$ and $\tau(s) = \bar{\tau}(s)$, $s \in I$. Let t_0, n_0, b_0 and $\bar{t}_0, \bar{n}_0, \bar{b}_0$ be the Frenet trihedrons at $s = s_0 \in I$ of α and $\bar{\alpha}$, respectively. Clearly, there is a rigid motion which takes $\bar{\alpha}(s_0)$ into $\alpha(s_0)$ and $\bar{t}_0, \bar{n}_0, \bar{b}_0$ into t_0, n_0, b_0. Thus, after performing this rigid motion on $\bar{\alpha}$, we have that $\bar{\alpha}(s_0) = \alpha(s_0)$.

By using the Frenet equations for both Frenet trihedrons $t(s), n(s), b(s)$ and $\bar{t}(s), \bar{n}(s), \bar{b}(s)$ with the conditions $t(s_0) = \bar{t}(s_0)$, $n(s_0) = \bar{n}(s_0)$, $b(s_0) = \breve{b}(s_0)$, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{ds}\{|t(s) - \bar{t}(s)|^2 + |n(s) - \bar{n}(s)|^2 + |b(s) - \bar{b}(s)|^2\} \\
= \ & \langle t(s) - \bar{t}(s), t'(s) - \bar{t}'(s)\rangle + \langle b(s) - \bar{b}(s), b'(s) - \bar{b}'(s)\rangle + \langle n(s) - \bar{n}(s), n'(s) - \bar{n}'(s)\rangle \\
= \ & k(s)\langle t(s) - \bar{t}(s), n(s) - \bar{n}(s)\rangle + \tau(s)\langle b(s) - \bar{b}(s), n(s) - \bar{n}(s)\rangle \\
& -k(s)\langle n(s) - \bar{n}(s), t(s) - \bar{t}(s)\rangle - \tau(s)\langle n(s) - \bar{n}(s), b(s) - \bar{b}(s)\rangle \\
= \ & 0 \quad \text{for all } s \in I
\end{aligned}$$

Thus, the above expression is constant, and, since it is zero for $s = s_0$, it is identically zero. It follows that $t(s) = \bar{t}(s)$, $n(s) = \bar{n}(s)$, $b(s) = \bar{b}(s)$ for all $s \in I$. Since

$$\frac{d\alpha}{ds}(s) = t(s) = \bar{t}(s) = \frac{d\bar{\alpha}}{ds}(s) \implies \frac{d(\alpha - \bar{\alpha})}{ds}(s) = 0 \quad \text{for all } s \in I \implies \alpha(s) = \bar{\alpha}(s) + a,$$

where a is a constant vector. Since $\alpha(s_0) = \bar{\alpha}(s_0)$, we have $a = 0$; hence, $\alpha(s) = \bar{\alpha}(s)$ for all $s \in I$.

The Local Canonical Form Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length without singular points of order 1. We shall write the equations of the curve, in a neighborhood of s_0, using the trihedron $t(s_0), n(s_0), b(s_0)$ as a basis for $\mathbb{R}^3$. We may assume, without loss of generality, that $s_0 = 0$, and we shall consider the (finite) Taylor expansion

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + R(s), \quad \text{where } \lim_{s\to 0}\frac{R(s)}{s^3} = 0.$$

Since $\alpha'(0) = t(0)$, $\alpha''(0) = k(0)\,n(0)$ and

$$\alpha'''(0) = (kn)'(0) = k'(0)\,n(0) + k(0)\,n'(0) = k'(0)\,n(0) - k^2(0)\,t(0) - k(0)\tau(0)\,b(0),$$

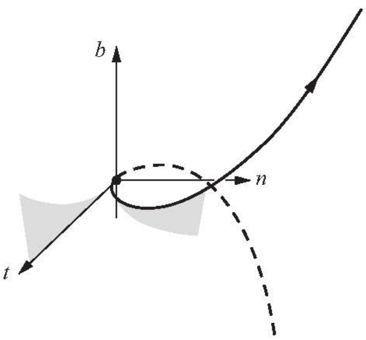
we obtain

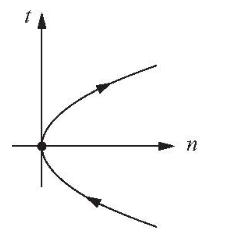
$$\alpha(s) - \alpha(0) = \left(s - \frac{s^3k^2(0)}{3!}\right)t(0) + \left(\frac{s^2k(0)}{2} + \frac{s^3k'(0)}{3!}\right)n(0) - \frac{s^3}{3!}k(0)\tau(0)b(0) + R(s).$$

Let us now take the system $Oxyz$ in such a way that the origin O agrees with $\alpha(0)$ and that $t(0) = (1,0,0)$, $n(0) = (0,1,0)$, $b(0) = (0,0,1)$. Under these conditions, the local canonical form of $\alpha(s) = (x(s), y(s), z(s))$, in a neighborhood of $s = 0$ is given by

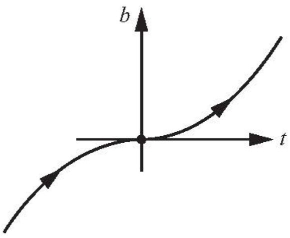
$$\begin{aligned} x(s) &= s - \frac{s^3k^2(0)}{6} + R_x, \\ y(s) &= \frac{s^2k(0)}{2} + \frac{s^3k'(0)}{6} + R_y, \\ z(s) &= -\frac{s^3k(0)\tau(0)}{6} + R_z, \end{aligned}$$

where $R(s) = (R_x, R_y, R_z)$.

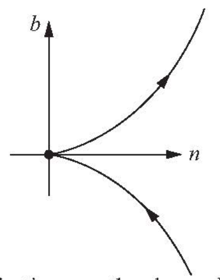




Projection over the plane *t n*



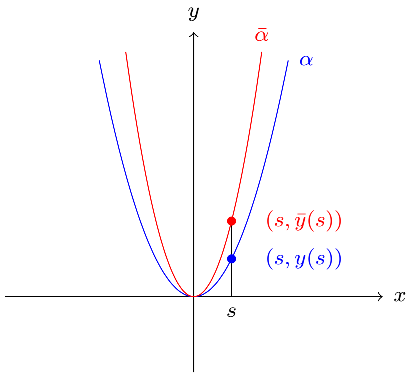
Projection over the plane *t b*



Projection over the plane *n b*

Example Using the local canonical form for $\bar{\alpha}$ and α, we have

$$\bar{k}(0) = \lim_{s\to 0}\frac{2\bar{y}(s)}{s^2} \geq \lim_{s\to 0}\frac{2y(s)}{s^2} = k(0).$$



The Isoperimetric Inequality

This is perhaps the oldest global theorem in differential geometry and is related to the following (isoperimetric) problem. Of all simple closed curves in the plane with a given length ℓ, which one bounds the largest area? In this form, the problem was known to the Greeks, who also knew the solution, namely, the circle.

Definitions A closed plane curve is a regular parametrized plane curve $\alpha : [a, b] \to \mathbb{R}^2$ such that α and all its derivatives agree at a and b; that is,

$$\alpha(a) = \alpha(b), \quad \alpha'(a) = \alpha'(b), \quad \alpha''(a) = \alpha''(b), \ldots$$

The curve α is simple if it has no further self-intersections; that is, if $t_1, t_2 \in [a, b)$, $t_1 \neq t_2$, then $\alpha(t_1) \neq \alpha(t_2)$.

We usually consider the curve $\alpha : [0, \ell] \to \mathbb{R}^2$ parametrized by arc length s; hence, ℓ is the length of α. Sometimes we refer to a simple closed curve C, meaning the trace of such an object.

We assume that a simple closed curve C in the plane bounds a region of this plane that is called the interior of C. This is part of the so-called Jordan curve theorem, which does not hold, for instance, for simple curves on a torus (the surface of a doughnut). Whenever we speak of the area bounded by a simple closed curve C, we mean the area of the interior of C. We assume further that the parameter of a simple closed curve can be so chosen that if one is going along the curve in the direction of increasing parameters, then the interior of the curve remains to the left. Such a curve will be called positively oriented.

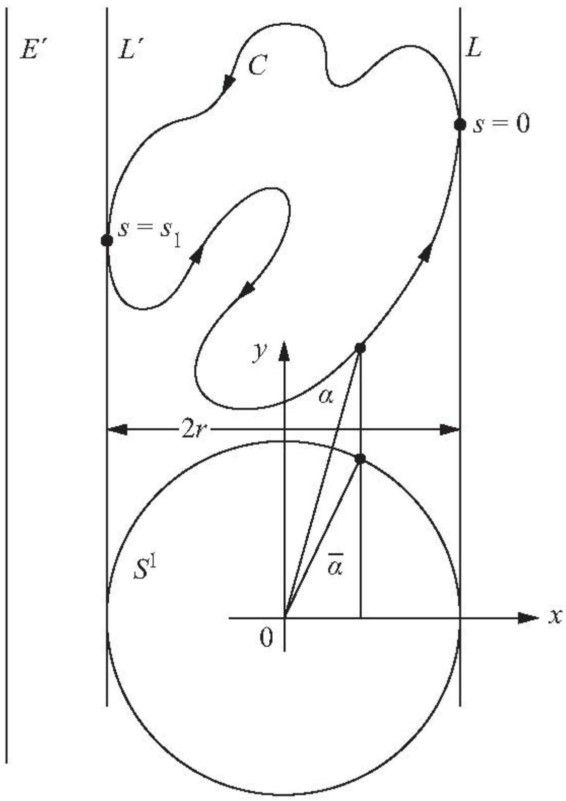
The Isoperimetric Inequality Let C be a simple closed plane curve with length ℓ, and let A be the area of the region bounded by C. Then

$$\ell^2 - 4\pi A \geq 0, \quad \text{and equality holds if and only if } C \text{ is a circle.}$$

Proof Let E and E' be two parallel lines which do not meet the closed curve C, and move them together until they first meet C. We thus obtain two parallel tangent lines to C, L and L', so that the curve is entirely contained in the strip bounded by L and L'. Consider a circle S^1 which is tangent to both L and L' and does not meet C. Let O be the center of S^1 and take a coordinate system with origin at O and the x axis perpendicular to L and L'. Parametrize C by arc length, $\alpha(s) = (x(s), y(s))$, so that it is positively oriented and the tangency points of L and L' are $s = 0$ and $s = s_1$, respectively.

We can assume that the equation of S^1 is

$$\bar{\alpha}(s) = (\bar{x}(s), \bar{y}(s)) = (x(s), \bar{y}(s)), \quad s \in [0, \ell].$$



Let $2r$ be the distance between L and L'. By using the Green's Theorem and denoting by $\bar{A}$ the area bounded by S^1, we have

$$A = \int_0^\ell xy'\, ds, \quad \bar{A} = \pi r^2 = -\int_0^\ell \bar{y}x'\, ds.$$

Thus,

$$\begin{aligned} A + \pi r^2 &= \int_0^\ell (xy' - \bar{y}x')\, ds \overset{(*)}{\leq} \int_0^\ell \sqrt{(xy' - \bar{y}x')^2}\, ds \\ &\overset{(**)}{\leq} \int_0^\ell \sqrt{(x^2 + \bar{y}^2)((x')^2 + (y')^2)}\, ds = \int_0^\ell \sqrt{(\bar{x}^2 + \bar{y}^2)}\, ds \\ &= \ell r, \end{aligned}$$

where we used that the inner product of two vectors v_1 and v_2 satisfies

$$|(v_1 \cdot v_2)|^2 \leq |(v_1)|^2 |(v_2)|^2.$$

We now notice the fact that the geometric mean of two positive numbers is smaller than or equal to their arithmetic mean, and equality holds if and only if they are equal. It follows that

$$\sqrt{A}\,\sqrt{\pi r^2} \overset{(\dagger)}{\leq} \frac{1}{2}(A + \pi r^2) \leq \frac{1}{2}\ell r \implies 4\pi A r^2 \leq \ell^2 r^2 \implies 4\pi A \leq \ell^2.$$

Suppose that the equality $4\pi A = \ell^2$ holds. Then equality must hold everywhere in $(*)$, $(**)$ and $(\dagger)$.

From the equality in $(\dagger)$, we have $A = \pi r^2$, $\ell = 2\pi r$ and r does not depend on the choice of the direction of L.

Furthermore, equality in $(*)$ and $(**)$ imply that

$$(x, \bar{y}) = \lambda(y', -x')$$

that is,

$$\lambda = \frac{x}{y'} = \frac{\bar{y}}{-x'} = \frac{\pm\sqrt{x^2 + \bar{y}^2}}{\sqrt{(y')^2 + (x')^2}} = \pm r.$$

Thus, we obtain that $x = \pm r y'$.

Since r does not depend on the choice of the direction of L, by using a counterclockwise rotation of $\pi/2$ and a translation of xy coordinates $(\tilde{x}, \tilde{y}) = (y - y_0, -x - x_0)$, we obtain

$$\tilde{x} = \pm r\tilde{y}' \iff y - y_0 = \mp r x'.$$

Thus,

$$x^2 + (y - y_0)^2 = r^2((x')^2 + (y')^2) = r^2$$

and C is a circle, as we wished.

Remarks

1. It is easily checked that the above proof can be applied to C^1 curves, that is, curves $\alpha(t) = (x(t), y(t))$, $t \in [a, b]$, for which we require only that the functions $x(t)$, $y(t)$ have continuous first derivatives (which, of course, agree at a and b if the curve is closed).
2. The isoperimetric inequality holds true for a wide class of curves. Direct proofs have been found that work as long as we can define arc length and area for the curves under consideration. For the applications, it is convenient to remark that the theorem holds for piecewise C^1 curves, that is, continuous curves that are made up by a finite number of C^1 arcs. These curves can have a finite number of corners, where the tangent is discontinuous.